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EFFECT OF PULSE SHAPE AND DISTRIBUTION ON THE PLASTIC DEFORMATION OF A CIRCULAR PLATE

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Abstract—The dynamic plastic deformation of a simply-supported circular plate subjected to a pressure distribution, which is an arbitrary function of radius and time, is found to depend on the moment history applied by the pressure and the moment history applied by the pressure excess over a conical distribution. Each of these moment histories can be replaced by an equivalent rectangular pulse in determining the final deformation at the center of the plate.

1. INTRODUCTION

A simply-supported circular plate, made of a rigid, perfectly-plastic material, is considered to be loaded by a pressure distribution which varies with both radial position and time. A closed-form solution is obtained for dynamic deformations which result solely from a basic velocity mode and those which also involve a hinge band response. The plastic deformation for the first case is found to depend on the pulse shape of the moment which the pressure distribution applies about the edge of the plate. However, the final deformation depends only on an equivalent rectangular moment pulse. This equivalent pulse has the same area and centroid as the arbitrarily defined moment loading but.does not depend on details of the loading history.

Conical pressure distributions, that is, those that decrease linearly with radial position, result in only basic mode responses. Pressure distributions which lie above the conical shape may activate a hinge band response also. The final plastic deformation is found to be determined by both the applied moment history and the history of the pressure excess over the conical shape. The applied moment history can be replaced again by its equivalent rectangular pulse and an analogously defined equivalent rectangular pulse can be used to characterize the moment produced by the excess pressure; these two equivalent rectangular pulses determine the final plastic deformation at the center of the plate.

Similarities in the forms of the solutions for the basic velocity mode and hinge band indicate the final plastic deformation at the center of the plate for pressure distributions which produce a hinge band response is exactly equal to the deformation which would be produced by the entire pressure distribution if it acted only in the basic mode minus the deformation which would be produced by the excess pressure acting alone on the plate in the basic mode.

Mode approximation techniques (Martin and Symonds, 1966) are being developed to take advantage of the simplifications available in rigid, perfectly-plastic analysis methods to obtain approximations to solutions for more realistic material behavior; e.g. see Kaliszky (1970), Jones and Wierzbicki (1976), Symonds and Chon (1978), Jones and Guedes Soares (1978), Symonds and Wierzbicki (1979), Symonds (1980a,b), Lepik (1982), Raphanel and Symonds (1984), Trossbach and Martin (1985), and Symonds and Mosquera (1985). A general feature of rigid, perfectly-plastic solutions, as discussed in Symonds and Fleming (1984), is a transient phase with velocity patterns involving travelling hinges or hinge bands, followed by a one-degree-of-freedom modal response. The results of this paper provide a means of replacing arbitrary pressure distributions and histories acting on a circular plate with rectangular pulses which eliminate the transient phase and activate only the simple modal behavior. Although the deformation profiles are not identical, the equivalent loading produces the same final displacement at the center of the plate as the more general loading.

The closed-form solution obtained here for the time-dependent velocity and displacement profiles produced by arbitrary dynamic loadings should be useful in validating computer programs having a dynamic plasticity capability. Moreover, although the assumption of rigid, perfectly-plastic material behavior restricts the range of direct applicability of the results, the equivalent rectangular pulse concept may provide a basis for extending the analysis to more general constitutive relations and for correlating experimental results.

Hopkins and Prager (1954) solved the problem of a simply-supported plate subjected to a uniform pressure applied as a rectangular pulse in time. For uniform pressure distributions, Perzyna (1958) derived a closed-form solution for arbitrary pulse shapes which produce a basic mode response and obtained numerical solutions to the governing differential equations for two pulse shapes that activated the hinge band response. Youngdahl (1971) obtained a closed-form solution for the hinge band response and showed that the influence of pulse shape on the final plastic deformation produced by both response patterns could be effectively eliminated by expressing the results in terms of an equivalent rectangular pressure pulse. This paper extends the closed-form solutions obtained previously to include axisymmetric pressure distributions which vary arbitrarily with both position and time. The pressure need not be separable into the product of a function of position and a function of time, and it is assumed to cause only the basic mode and hinge band response.

Florence (1966, 1977), Conroy (1969), and Krajcinovic (1972) obtained solutions for rigid, perfectly-plastic plates loaded uniformly over a central region by pressure pulses, and Lepik (1974) and Youngdahl and Krajcinovic (1986) considered infinite plates subjected to dynamic loading.

2. STATEMENT OF PROBLEM

Consider a simply-supported circular plate of radius R subjected to a dynamic pressure pulse $P(r, t)$, where r is the radial coordinate and t is time. Under the usual assumptions of small deflection theory of thin plates, the equations of motion are

$$\frac{\partial^2(rM_r)}{\partial r^2} - \frac{\partial M_\theta}{\partial r} = \mu r \frac{\partial V}{\partial t} - rP, \qquad V = \frac{\partial W}{\partial t} \tag{1}$$

where $M_r(r, t)$ and $M_\theta(r, t)$ are the radial bending moment and circumferential bending moment per unit arc length, respectively, μ is the mass per unit surface area, and $V(r, t)$ and $W(r, t)$ are the lateral velocity and deflection. The pressure distribution is assumed to be between the associated conical and uniform pressure distributions at every time t; i.e.

$$P(0, t)\left(1 - \frac{r}{R}\right) \leqslant P(r, t) \leqslant P(0, t). \tag{2}$$

This assumption assures that the bending moment is positive throughout the plate; this in turn eliminates the possibility of the bi-linear velocity mode which may occur for loadings which are more concentrated toward the center of the plate.

The material of the plate is rigid, perfectly-plastic, and insensitive to strain rate. Either the Tresca hexagon or the Johansen square can be used as the yield condition since only the first quadrant, where they coincide, is relevant.

When the limit load of the plate is exceeded at time t_i, a plastic hinge forms at the center and the plate begins to deform in a conical velocity mode, which will be referred to as the basic mode. The boundary and initial conditions are

$$\begin{aligned} &M_r(0, t) = M_0, \quad M_r(R, t) = 0, \quad M_\theta(r, t) = M_0, \\ &V(r, t_i) = 0, \quad W(r, t_i) = 0 \end{aligned} \tag{3}$$

where M_0 is the bending moment at yield.

For some loadings, the central hinge spreads into a hinge band having radius $\rho(t)$. The boundary condition at the center of the plate is then replaced by

$$M_r = M_0, \quad 0 \leqslant r \leqslant \rho(t), \qquad \frac{\partial M_r}{\partial r} = 0 \quad \text{at} \quad r = \rho(t) \tag{4}$$

where the specification of the vanishing derivative follows from the restrictions on discontinuities at a moving hinge given by Hopkins and Prager (1954).

3. SOLUTION FOR BASIC MODE

The flow rule associated with the yield condition implies that $\partial^2 V/\partial r^2 = 0$ for $0 < r \leqslant R$, so that we can take the basic velocity mode

$$V(r, t) = v(t)\frac{R-r}{R} \tag{5}$$

where $v(t)$ is the velocity at the center of the plate. The solution to the differential equations, eqns (1), using the boundary and initial conditions (3), is then

$$\begin{aligned} v(t) &= \frac{12}{\mu R^3}\int_{t_i}^{t} [H(t')-H_y]\,\mathrm{d}t', \\ w(t) &= \frac{12}{\mu R^3}\int_{t_i}^{t} (t-t')[H(t')-H_y]\,\mathrm{d}t', \\ W(r,t) &= \frac{R-r}{R}w(t), \\ M_r(r,t) &= \frac{(R^3-2Rr^2+r^3)}{R^4}[H_y - H^*(t)] + \frac{h^*(r,t)}{R} \\ &\quad + \frac{R-r}{Rr}\int_0^r (r')^2 P^*(r',t)\,\mathrm{d}r'. \end{aligned} \tag{6}$$

In the above, $H(t)$ is the moment per unit angle applied about the edge of the plate by the pressure distribution, $H_y = RM_0$ is the yield value of this moment, P^* is the *excess* pressure distribution after the associated conical distribution is subtracted from the applied pressure, H^* is the applied moment produced by P^* about the edge of the plate, and h^* is the applied moment produced by the portion of P^* between r and R; i.e.

$$\begin{aligned} H(t) &= \int_0^R r(R-r)P(r,t)\,\mathrm{d}r, \\ P^*(r,t) &= P(r,t) - P(0,t)\left(\frac{R-r}{R}\right), \\ H^*(t) &= \int_0^R r(R-r)P^*(r,t)\,\mathrm{d}r, \\ h^*(r,t) &= \int_r^R r'(R-r')P^*(r',t)\,\mathrm{d}r'. \end{aligned} \tag{7}$$

The motion begins at time t_i such that

$$H(t_i) = H_y. \tag{8}$$

It ends at time t_f when $V = 0$. From the first of eqns (6) we have

$$\int_{t_i}^{t_f} [H(t) - H_y] \, dt = 0 \tag{9}$$

which implies that the average applied moment during the deformation is the applied moment that initiates yielding.

Let I be the impulse (per unit angle) applied by the pulse $H(t)$ during the deformation, and let t_c be the interval between t_i and the centroid of the pulse; i.e.

$$I = \int_{t_i}^{t_f} H(t) \, dt, \qquad t_c = \frac{1}{I}\int_{t_i}^{t_f} (t - t_i) H(t) \, dt. \tag{10}$$

The final plastic deformation at the center of the plate is then, using eqns (6), (9), and (10)

$$W(0, t_f) = \frac{6I}{\mu R^3 H_y}(I - 2H_y t_c). \tag{11}$$

Define an *equivalent* rectangular pulse of amplitude H_e and duration T_e such that it has the same impulse and centroid as $H(t)$. Then

$$t_e = 2t_c, \qquad H_e = \frac{I}{t_e}. \tag{12}$$

It can readily be shown that $H_e > H_y$ if $H(t) > H_y$ over a time interval starting at t_i and eqn (9) holds. In terms of H_e and t_e, eqn (11) becomes

$$W(0, t_f) = \frac{6H_e^2 t_e^2}{\mu R^3 H_y}\left(1 - \frac{H_y}{H_e}\right). \tag{13}$$

For the basic deformation mode, the final plastic deformation produced by the applied moment history $H(t)$ is exactly equal to the deformation produced by a rectangular pulse having the same area and centroid. In particular, $P(r, t)$ need not be expressible as the product of a function of position and a function of time, which implies that the shape of the pressure distribution can vary with time and the correlation will still be exact.

A hinge band does not form at the center of the plate if M_r is a relative maximum there. Since $M_r = M_0$ and $\partial M/\partial r = 0$ at $r = 0$, the condition for no hinge band formation is $\partial^2 M_r/\partial r^2 < 0$ at $r = 0$; using eqns (6) and (7), we have

$$\frac{\partial^2 M_r}{\partial r^2} = \frac{4}{R^3}[H^*(t) - H_y] \quad \text{at} \quad r = 0. \tag{14}$$

Consequently, the motion is entirely in the basic velocity mode if the maximum value of $H^*(t)$ is less than H_y. In particular, a hinge band cannot occur if $P(r, t)$ is always conical since $H^* = 0$ then.

4. SOLUTION FOR HINGE BAND MODE

Consider loadings for which $H^*(t)$ exceeds H_y over some time interval. A hinge band begins to form at the center of the plate at time t_i^*, such that

$$H^*(t_i^*) = H_y \tag{15}$$

and grows to occupy the region $0 \leqslant r \leqslant \rho(t)$. The band reaches its greatest extent at time t_m; it then decreases until time t_f^* such that $\rho(t_f^*) = 0$. The plate motion then reverts to the basic velocity mode.

The solution given by eqns (5) and (6) holds for $t_i \leqslant t \leqslant t_i^*$. Since $M_r = M_\theta = M_0$ throughout the hinge band, the solution to eqn (1) becomes

$$V(r,t) = \frac{1}{\mu}\int_{t_i^*}^{t} P^*(r,t')\,dt' + \left(\frac{R-r}{R}\right)[v(t)-v^*(t)] + V_b(r) \tag{16}$$

for $t_i^* \leqslant t \leqslant t_f^*$ and $0 \leqslant r \leqslant \rho(t)$. In the above, $v(t)$ is given by the first of eqns (6); $v^*(t)$ is defined by

$$v^*(t) = \frac{12}{\mu R^3}\int_{t_i^*}^{t} [H^*(t')-H_y]\,dt'; \tag{17}$$

the identity

$$R^4[P(r,t)-P^*(r,t)] = 12(R-r)[H(t)-H^*(t)] \tag{18}$$

has been used so that subsequent equations will be simpler; and $V_b(r)$ is a function to be determined from continuity of velocity at $\rho(t)$.

The flow rule implies that $\partial^2 V/\partial r^2 = 0$ outside of the hinge band. Consequently, we will take

$$V(r,t) = V_\rho(t)\frac{R-r}{R-\rho(t)}, \qquad \rho(t) \leqslant r \leqslant R \tag{19}$$

such that $V(R,t) = 0$ and $V_\rho(t)$ is the lateral velocity at $r = \rho$. Substitution into eqn (1), integration, and use of the boundary conditions then gives

$$\frac{d}{dt}\left(\frac{V_\rho}{R-\rho}\right) = \frac{12[h^*(\rho,t)-H_y]}{\mu(R-\rho)^3(R+3\rho)} + \frac{12[H(t)-H^*(t)]}{\mu R^4} \tag{20}$$

and

$$M_r = \frac{(R-r)(R^3r+R^2r^2-Rr^3-4R\rho^3+3\rho^4)}{rR(R-\rho)^3(R+3\rho)}[H_y-h^*(\rho,t)] + \frac{h^*(r,t)}{R} + \frac{R-r}{Rr}\int_\rho^r (r')^2P^*(r',t)\,dr' \tag{21}$$

for $t_i^* \leqslant t \leqslant t_f^*$.

While the hinge band is growing, $\partial^2 M_r/\partial r^2 = 0$ at $r = \rho$. Using eqn (21), this is equivalent to

$$12[h^*(\rho,t)-H_y]-(R-\rho)^2(R+3\rho)P^*(\rho,t)=0 \tag{22}$$

which gives the relation between ρ and t for $t_i^* \leqslant t \leqslant t_m$. Substitution of eqn (22) into eqn (20), integration, and the use of $\rho(t_i^*)=0$ and $V_\rho(t_i^*)=v(t_i^*)$ gives

$$V_\rho(t)=\frac{R-\rho}{R}\left[\frac{R}{\mu}\int_{t_i^*}^{t}\frac{P^*(\rho',t')}{R-\rho'}\,\mathrm{d}t'+v(t)-v^*(t)\right] \tag{23}$$

where $\rho'=\rho(t')$ and $t_i^* \leqslant t \leqslant t_m$. From eqn (19)

$$V(r,t)=\frac{R-r}{R}\left[\frac{R}{\mu}\int_{t_i^*}^{t}\frac{P^*(\rho',t')}{R-\rho'}\,\mathrm{d}t'+v(t)-v^*(t)\right] \tag{24}$$

for $\rho \leqslant r \leqslant R$ and $t_i^* \leqslant t \leqslant t_m$. Imposing continuity of velocity at $r=\rho$ then gives

$$V_b(r)=\frac{R-r}{\mu}\int_{t_i^*}^{\tau(r)}\frac{P^*(\rho',t')}{R-\rho'}\,\mathrm{d}t'-\frac{1}{\mu}\int_{t_i^*}^{\tau(r)}P^*(r,t')\,\mathrm{d}t' \tag{25}$$

$$V(r,t)=\frac{R-r}{R}[v(t)-v^*(t)]+\frac{R-r}{\mu}\int_{t_i^*}^{\tau(r)}\frac{P^*(\rho',t')}{R-\rho'}\,\mathrm{d}t'+\frac{1}{\mu}\int_{\tau(r)}^{t}P^*(r,t')\,\mathrm{d}t' \tag{26}$$

for $0 \leqslant r \leqslant \rho$. The time $\tau(r)$ is when $\rho=r$ for $t_i^* \leqslant \tau \leqslant t_m$ and is found from eqn (22).

Equations (16)–(27) remain applicable for the time interval $t_m \leqslant t \leqslant t_f^*$ when the hinge band is shrinking. The function $V_b(r)$ is now known for every position $r \leqslant \rho$ within the band so eqn (26) is valid for $t_m \leqslant t \leqslant t_f^*$. We must still determine $\rho(t)$ for this interval and $V(r,t)$ outside the hinge band.

Setting $r=\rho$ in eqn (26) gives $V_\rho(t)$ for $t_m \leqslant t \leqslant t_f^*$, and substitution into eqn (20) gives a differential equation for $\rho(t)$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}\left[\int_{\tau(\rho)}^{t}P^*(\rho,t')\,\mathrm{d}t'+(R-\rho)\int_{\tau(\rho)}^{t}\frac{\partial P^*(\rho,t')}{\partial\rho}\,\mathrm{d}t'\right]+(R-\rho)P^*(\rho,t)+\frac{12[H_y-h^*(\rho,t)]}{(R-\rho)(R+3\rho)}=0. \tag{27}$$

Using eqn (22), the solution is

$$12\int_{\tau(\rho)}^{t}[h^*(\rho,t')-H_y]\,\mathrm{d}t'-(R-\rho)^2(R+3\rho)\int_{\tau(\rho)}^{t}P^*(\rho,t')\,\mathrm{d}t'=0 \tag{28}$$

which relates ρ and t for $t_m \leqslant t \leqslant t_f^*$.

At t_f^*, $\rho=0$; since $\tau(0)=t_i^*$, $P^*(0,t)=0$, and $h^*(0,t')=H^*(t')$, eqn (28) gives

$$\int_{t_i^*}^{t_f^*}[H^*(t')-H_y]\,\mathrm{d}t'=0 \tag{29}$$

which determines t_f^*. This indicates that the average value of $H^*(t)$ during the hinge band mode is H_y.

The velocity outside the hinge band is, using eqn (19)

$$V(r,t) = \frac{R-r}{R}[v(t)-v^*(t)] + \frac{R-r}{\mu}\int_{t_i^*}^{\tau(\rho)} \frac{P^*(\rho',t')}{R-\rho'}\,dt' + \frac{R-r}{\mu(R-\rho)}\int_{\tau(\rho)}^{t} P^*(\rho,t')\,dt' \tag{30}$$

for $\rho \leqslant r \leqslant R$ and $t_m \leqslant t \leqslant t_f^*$. The results for $W(r,t)$ for the hinge band mode are given in the Appendix.

The motion reverts to the basic mode for $t_f^* \leqslant t \leqslant t_f$.

Let $w^*(t)$ be defined by

$$w^*(t) = \frac{12}{\mu R^3}\int_{t_i^*}^{t} (t-t')\,[H^*(t')-H_y]\,dt' \tag{31}$$

and let $w(t)$ be given by the second of eqns (6). For problems where $H^*(t) > H_y$ over some time interval, the velocity and displacement at the plate center are then

$$\begin{aligned} &V(0,t) = v(t), \quad W(0,t) = w(t) \quad \text{for} \quad t_i \leqslant t \leqslant t_i^*;\\ &V(0,t) = v(t)-v^*(t), \quad W(0,t) = w(t)-w^*(t) \quad \text{for} \quad t_i^* \leqslant t \leqslant t_f^*;\\ &V(0,t) = v(t), \quad W(0,t) = w(t)-w^*(t_f^*) \quad \text{for} \quad t_f^* \leqslant t \leqslant t_f. \end{aligned} \tag{32}$$

Consequently, the time t_f when the motion stops is given by eqn (9) again.

Let I^* be the impulse per unit angle associated with $H^*(t)$ during the hinge band existence, and let t_c^* locate the centroid of H^* between t_i^* and t_f^*; that is

$$\begin{aligned} I^* &= \int_{t_i^*}^{t_f^*} H^*(t)\,dt,\\ t_c^* &= \frac{1}{I^*}\int_{t_i^*}^{t_f^*} (t-t_i^*)H^*(t)\,dt. \end{aligned} \tag{33}$$

Let H_e^* and t_e^*, the amplitude and duration of the equivalent rectangular pulse associated with $H^*(t)$, be defined by

$$t_e^* = 2t_c^*, \qquad H_e^* = \frac{I^*}{t_e^*}. \tag{34}$$

As before, $H_e^* > H_y$ if $H^*(t) > H_y$ over some time interval.

The final plastic deformation at the center of the plate then becomes, using eqns (6), (9), (10), (12), (29), and (32)–(34)

$$W(0,t_f) = \frac{6}{\mu R^3 H_y}\left[(H_e t_e)^2\left(1-\frac{H_y}{H_e}\right) - (H_e^* t_e^*)^2\left(1-\frac{H_y}{H_e^*}\right)\right] \tag{35}$$

if $H_e \geqslant H_y$ and $H_e^* \geqslant H_y$.

5. SUMMARY AND CONCLUSIONS

A solution has been obtained for the dynamic plastic deformation of a simply-supported circular plate made of a rigid, perfectly-plastic material. The applied pressure $P(r,t)$ need not be separable into the product of a function of position and a function of time but is assumed to lie between a conical and a uniform distribution at all times. The motion starts at time t_i when the applied moment $H(t)$ about the edge reaches the yield value H_y and stops at time t_f when the area under the $H(t)$ pulse between t_i and t_f equals the area

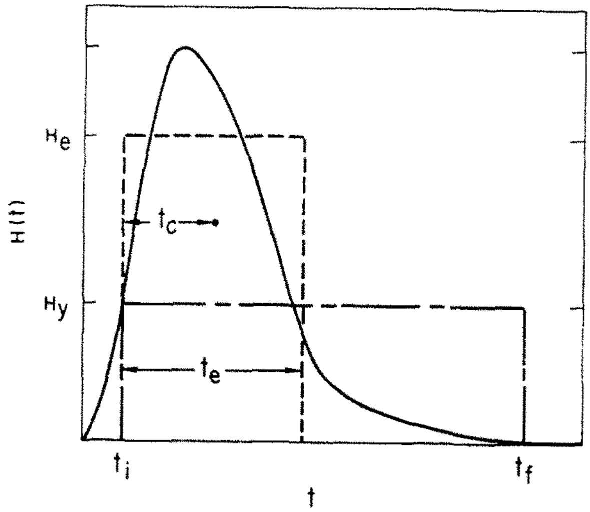

Fig. 1. Relations between $H(t)$, H_y, and H_e.

under H_y over this interval (see eqn (9) and Fig. 1). The motion is entirely in the basic mode if $H^*(t)$, the applied moment produced by the pressure excess over the associated conical distribution, is always less than H_y. The final plastic deformation then is conical and can be expressed in terms of an equivalent pulse of amplitude H_e and duration t_e which has the same impulse and centroid as $H(t)$.

Deformation in a hinge band mode occurs if $H^*(t) \geqslant H_y$ in an interval beginning at time t_i^* such that $H^*(t_i) = H_y$. The hinge band radius $\rho(t)$ is given by eqn (22) while the band is growing and by eqn (28) while it is shrinking. These equations must be solved numerically for most loadings. However, it is unnecessary to determine $\rho(t)$ if only the final deformation at the center of the plate is needed since $W(0, t_f)$ depends only on equivalent rectangular pulses associated with $H(t)$ and $H^*(t)$.

As a specific example, consider $P(r, t)$ given by

$$P(r,t) = P_0 \frac{4t(t_0-t)}{t_0^2}\left\{1+[b(t)-1]\frac{r}{R}\right\}, \quad 0 \leqslant t \leqslant t_0; \qquad (36)$$
$$= 0, t > t_0$$

so that the pressure at the center of the plate is parabolic, reaching its maximum value of P_0 at $t = t_0/2$. The spatial distribution during the pulse is determined by $b(t)$. Figure 2 shows $H(t)$, H_e, $H^*(t)$, and H_e^* for $P_0 = 48H_y/R^3$ and $b(t) = (t-t_i)/(t_0-t_i)$; i.e. the pressure distribution is conical when deformation begins at time t_i and becomes uniform by the end of the pulse. The final deformation shape for this loading is shown in Fig. 3, along with results for the conical pressure distribution given by $b(t) = 0, 0 \leqslant t \leqslant t_0$, and for the uniform pressure loading given by $b(t) = 1$, $0 \leqslant t \leqslant t_0$. The value of P_0 is the same for all three curves.

The equations determining H^*, t_i^*, t_f^*, H_e^*, and t_e^* are identical to those for the corresponding unstarred quantities so that Fig. 1 pertains to the excess loading also. The forms of the solutions given by eqns (13) and (35) for loadings which activate the basic mode and those which cause hinge band formation and the analogies between the starred and unstarred quantities suggest a "subtraction" principle: (1) compute the deformation produced by $H(t)$ assuming only the basic mode is active; (2) apply $H^*(t)$ independently to the plate and compute the basic mode deformation it produces. The difference between these two solutions is identical to the exact solution at the center of the plate.

If $H_e > H_y$ and $H_e^* < H_y$, the result for step 2 is zero and the step 1 result is the correct solution for the loading. If $H_e > H_y$ and $H_e^* > H_y$, the result for step 1 overestimates the deflection at the center of the plate because the deformation in the hinge band region is

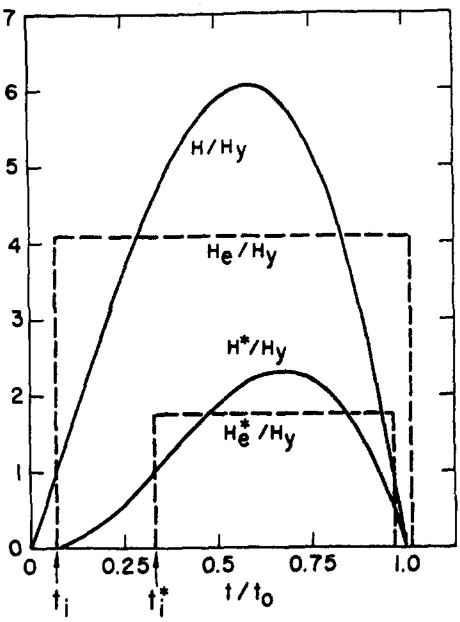

Fig. 2. $H(t)$, H_e, $H^*(t)$, and H_e^* for example problem.

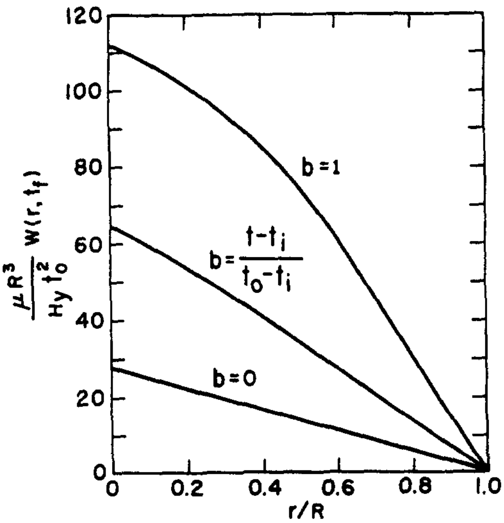

Fig. 3. Final deformation shapes for example problems.

more rounded than the basic mode predicts; the amount of this overestimation is exactly the central deflection which the excess load $P^*(r, t)$ would produce if acting alone. Subtracting the two conical deformation shapes underestimates the deflection at other points, however (see Appendix for $W(r, t)$ for $H_e^* > H_y$).

This subtraction principle is a surprising result considering the nonlinearity of the problem and may be only a fortuitous coincidence. The standard superposition principle, adding the solution for the excess loading to the solution for the conical pressure distribution, significantly underestimates the central deflection and is not valid.

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APPENDIX

The displacement $W(r, t)$ for the hinge band mode is found by integrating the velocity distribution given by eqns (24), (26), and (30) and making use of eqns (22) and (28) for $\rho(t)$. Another relation which is needed is

$$12\int_{\tau(\rho)}^{t}\frac{[h^*(\rho',t')-H_y]}{(R-\rho')^3(R+3\rho')}\,\mathrm{d}t' = \frac{1}{R-\rho}\int_{\tau(\rho)}^{t}P^*(\rho,t')\,\mathrm{d}t' \tag{A1}$$

for $t_i^* \leqslant \tau \leqslant t_m$ and $t_m \leqslant t \leqslant t_f^*$, which can be verified by differentiating with respect to t and using eqns (22) and (27); as before, $\rho' = \rho(t')$.

The resulting expressions for $W(r, t)$ are

$$W(r,t) = \frac{R-r}{R}w(t) \quad \text{for} \quad t_i \leqslant t \leqslant t_i^*, \quad 0 \leqslant r \leqslant R;$$

$$W(r,t) = \frac{R-r}{R}[w(t)-w^*(t)] + \frac{1}{\mu}\int_{\tau(r)}^{t}(t-t')P^*(r,t')\,\mathrm{d}t' + \frac{R-r}{\mu}\int_{t_i^*}^{\tau(r)}\frac{(t-t')P^*(\rho',t')}{R-\rho'}\,\mathrm{d}t' \tag{A2}$$

for $t_i^* \leqslant t \leqslant t_f^*$, $0 \leqslant r \leqslant \rho(t)$;

$$W(r,t) = \frac{R-r}{R}[w(t)-w^*(t)] + \frac{(R-r)}{\mu}\int_{t_i^*}^{t}\frac{(t-t')P^*(\rho',t')}{R-\rho'}\,\mathrm{d}t'$$

for $t_i^* \leqslant t \leqslant t_m$, $\rho(t) \leqslant r \leqslant R$;

$$W(r,t) = \frac{R-r}{R}[w(t)-w^*(t)] + \frac{12(R-r)}{\mu}\int_{\beta(r)}^{t}\frac{(t-t')[h^*(\rho',t')-H_y]}{(R-\rho')^3(R+3\rho')}\,dt' + \frac{1}{\mu}\int_{\tau(r)}^{\beta(r)}(t-t')P^*(r,t')\,dt' + \frac{R-r}{\mu}\int_{t_i^*}^{\tau(r)}\frac{(t-t')P^*(\rho',t')}{R-\rho'}\,dt'$$

for $t_m \leqslant t \leqslant t_f^*$, $\rho(t) \leqslant r \leqslant \rho(t_m)$;

$$W(r,t) = \frac{R-r}{R}[w(t)-w^*(t)] + \frac{12(R-r)}{\mu}\int_{t_i^*}^{t}\frac{(t-t')[h^*(\rho',t')-H_y]}{(R-\rho')^3(R+3\rho')}\,dt'$$

for $t_m \leqslant t \leqslant t_f^*$, $\rho(t_m) \leqslant r \leqslant R$;

$$W(r,t) = \frac{R-r}{R}[w(t)-w^*(t_f^*)] + \frac{12(R-r)}{\mu}\int_{\beta(r)}^{t_f^*}\frac{(t_f^*-t')[h^*(\rho',t')-H_y]}{(R-\rho')^3(R+3\rho')}\,dt' + \frac{1}{\mu}\int_{\tau(r)}^{\beta(r)}(t_f^*-t')P^*(r,t')\,dt' + \frac{R-r}{\mu}\int_{t_i^*}^{\tau(r)}\frac{(t_f^*-t')P^*(\rho',t')}{R-\rho'}\,dt'$$

for $t_f^* \leqslant t \leqslant t_f$, $0 \leqslant r \leqslant \rho(t_m)$;

$$W(r,t) = \frac{R-r}{R}[w(t)-w^*(t_f^*)] + \frac{12(R-r)}{\mu}\int_{t_i^*}^{t_f^*}\frac{(t_f^*-t')[h^*(\rho',t')-H_y]}{(R-\rho')^3(R+3\rho')}\,dt'$$

for $t_f^* \leqslant t \leqslant t_f$, $\rho(t_m) \leqslant r \leqslant R$.

In the above, $\tau(r)$ and $\beta(r)$ are the times at which $\rho = r$ when the hinge band is increasing and decreasing, respectively. Consequently, using eqn (22), $\tau(r)$ is the solution of

$$12[h^*(r,\tau)-H_y]-(R-r)^2(R+3r)P^*(r,\tau) = 0 \tag{A3}$$

for $t_i^* \leqslant \tau \leqslant t_m$, and, using eqn (28), $\beta(r)$ is the solution of

$$12\int_{\tau(r)}^{\beta}[h^*(r,t')-H_y]\,dt' - (R-r)^2(R+3r)\int_{\tau(r)}^{\beta}P^*(r,t')\,dt' = 0 \tag{A4}$$

for $t_m \leqslant \beta \leqslant t_f^*$.